Rays and Phases: A Paradox?*

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Dedicatory

It is an honor to be able to speak in this meeting celebrating George Sudarshan's birthday; to know him and to discuss physics with him has been a great privilege for me

The states of a quantum mechanical system are represented by *rays* in Hilbert space, but interference phenomena, Berry phase, etc. make reference to *vectors*. We show how to solve this apparent paradox by appropriate use of the *vector bundle structure* of quantum theory.

1. The *states* of a quantum mechanical system are represented by *rays*, i.e. one-dimensional subspaces of a Hilbert space; but the eigenvalue problems, the evolution, collision theory, interference phenomena etc. are all set in a *vector*-like formulation.

Are both pictures consistent? In particular are *phase differences*, typical of vector subtraction, and measurable in interference fringes, describable in terms of rays only? This is the question we address in this paper.

Let H be the Hilbert space of vectors, and $\bar{H} = \mathbf{C}P(\infty) = PH$ the projective space; the map $H \to \bar{H}$ takes the ray of a vector; it is enough to take normalized vectors mod a phase; in mathematical terms we have the bundles

$$\begin{array}{llll} \eta: & \mathbf{C}^* & \to H - \{0\} & \to \bar{H} \\ & \downarrow & & \\ l: & \mathbf{C} & \to E & \to \bar{H} \\ & \uparrow & & \\ \eta_0: & U(1) & \to S(\infty) & \to \bar{H}, \end{array} \quad (1)$$

where η is the definition of projective space, η_0 the restriction to compact fibre, $S(\infty)$ the (infinite dimensional) sphere of norm-one vectors, and l the associated line bundle. Finally the total space E is similar to H:

$$E - \{0 \text{ section}\} \approx H - \{0\} . \quad (2)$$

We shall see that the bundle formalism allows to use rays and vectors, as long as everything will be "fibre-preserving".

2. We describe now some consequences of the bundle framework. *First* physics is described as a geometry, *not* as algebra; physical states are "points", related to each other by transformations (symmetry); the projective space $\mathbf{C}P(\infty)$ is just such a geometrical structure; there is no "state zero", nor can the states be added, etc., as they were elements of a *vector* space, which is an algebraic structure.

Secondly, there is no *superposition principle*, in the ordinary sense that state (1) plus state (2) make state (3): you cannot add rays, as you cannot add points. We can think of two "correct" statements of what is usually called the "superposition principle":

1) States (1) and (2) define a (complex bi-)plane; any state (ray) in this plane can be considered as a "superposition" of states (1) and (2); with hindsight, this can be seen in Dirac's book on quantum mechanics.

2) Given a state, it has a well-defined projection in any other state, i.e. the squared modulus of the scalar product of any two normalized vector representatives; in particular, any state has projections in a complete orthonormal set of rays. In this sense one speaks more properly of *principle of decomposition* of states; this is how Jauch states the (superposition) principle.

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Of course the projections of a state ψ in a base e_i, namely $r_i = |\langle \psi | e_i \rangle|^2$ do not fix the state completely; for that one needs to know the relative phases. In fact the overlap between two rays φ, ψ,

$$R(\varphi, \psi) = \frac{\langle \varphi | \psi \rangle}{\| \varphi \|^2 \| \psi \|^2}, \tag{3}$$

endowes the state space with a metric; in the projective space $\mathbf{C}P^1 = S^2$ associated to the complex biplane spanned by $|\varphi\rangle$ and $|\psi\rangle$ we have the metric

$$\text{distance}(\varphi, \psi) = 1 - \operatorname{Tr} \varrho_\varphi \varrho_\psi = \sin^2(\Phi/2), \tag{4}$$

where ϱ are the projectors and Φ the angle they make in S^2, e.g. orthogonal states differ by 1; however, $\| |\psi\rangle - |\varphi\rangle \| = \sqrt{2}$, if $\langle \psi | \varphi \rangle = 0$; this $2 \to 1$ change is due to the Hopf bundle

$$\beta\colon S^1 \to S^3 \to S^2 \tag{5}$$

which should be understood as the "square root" or the "spinor" bundle of the tangent bundle to S^2.

As a *third consequence* of the bundle structure, we remark that wave functions are *not*, in general, to be identified with complex-valued functions; they are rather *sections* in the associated line bundle of (1). In particular they can be "multi-valued"; the distinction is due to the non-trivial nature of the bundles (1) and (5), and it is essential to understand the "multi-valuedness" which appears in the Aharonov-Bohm effect, the Berry phase, etc.

As a *fourth* consequence, in spite of the projective nature of state space one can work with *vectors* as long as the operations are bundle morphisms, i.e. preserve the fibres; that is, only *linear* (or antilinear) operators are allowed: this is the true nexus between the *geometric* character of the states and the *algebraic* nature of the operations in quantum physics.

This means that e.g. eigenvalues are *measurable*, because the spectral problem is linear,

$$\mathbf{R}|\psi\rangle = \lambda|\psi\rangle \Rightarrow A(C|\psi\rangle) = \lambda C|\psi\rangle. \tag{6}$$

If $A = A^+$, then $\lambda \in \mathbf{R}$, e.g. the energy eigenvalues are observable; if $A^+ = A^{-1}, |\lambda| = 1$, e.g. we measure *phases* as the Berry phase or the dynamical phase, when A is the evolution operator for a closed loop in state space. In this way a unified description of phases as *eigenvalues of unitary operators* is obtained; if one wishes a purely "projective" characterization of instantaneous measurements, one can write in the above formula

$$\lambda = \operatorname{Tr} A \varrho_\psi \quad \text{when} \quad [A, \varrho_\psi] = 0$$
$$\text{and} \quad \varrho_\psi = |\psi\rangle\langle\psi|. \tag{7}$$

3. As an example of *motion of the states* vs. *motion of the vectors*, let us look at the simplest quantum-mechanical system, namely a particle in quantum two space dimensions. We have a two-dimensional *real* vector space $\mathbf{R}^2$; a general state is a projector ϱ and the "hamiltonian" will be antisymmetric,

$$\varrho = \tfrac{1}{2}(1 + \sigma_3 x + \sigma_1 y); \quad H = -i\sigma_2 h;$$
$$x, y, h \in \mathbf{R}; \quad x^2 + y^2 = 1. \tag{8}$$

Taking $(x, y) = (\cos\varphi, \sin\varphi)$, ϱ can be written as $|\psi\rangle\langle\psi|$, where

$$\varrho = \begin{pmatrix} \cos^2\varphi & \cos\varphi\sin\varphi \\ \cos\varphi\sin\varphi & \sin^2\varphi \end{pmatrix},$$
$$|\psi\rangle = \pm\begin{pmatrix} \cos\varphi/2 \\ \sin\varphi/2 \end{pmatrix}. \tag{9}$$

Observe the ambiguity ($\pm$) and the "halfness" ($\varphi \to \varphi/2$). Now the evolution equations

$$\dot{\varrho} = [H, \varrho], \quad \varrho(0) = \varrho(\varphi), \quad |\dot{\psi}\rangle = H|\psi\rangle \tag{10}$$

will have the solution $\varphi(t) = \varphi + th/2$, and therefore *the states* perform a cyclic motion of period $T_0 = 4\pi/h$ but the *vectors* will have $T_1 = 8\pi/h = 2T_0$! In particular $|\psi(T_0)\rangle = -|\psi(T_1)\rangle$: this is the simplest case of *Berry phase*, namely $\pi = 180°$. The spinor or "square root" character of the *real* bundle

$$\alpha\colon O(1) = Z_2 \to S^1 \to S^1 = \mathbf{R}P^1 \tag{11}$$

is the equivalent to (5) in this simpler case; it is well known how the "magnetic problem" associated to (5) will give an angle as Berry phase.

4. To finalize, let us talk about *interferences*; it is here where the contrast between *rays* and *phases* is sharper.

In the typical two-slit experiment, let ψ_s be the state at the source, which is propagated to the screen P by two ways (1) and (2). Let $\langle\psi_s\rangle$ *be a representative vector*, and let $|\psi_p^1\rangle$ and $|\psi_p^2\rangle$ be the propagation along paths (1) and (2). The point is now the following: The *phase difference* of $|\psi_p^1\rangle$ and $|\psi_p^2\rangle$ is *observable* because it is a projective concept, i.e. if $|\bar{\psi}_s\rangle$ were another representative vector of state ψ_s, the phase difference between $|\bar{\psi}_p^1\rangle$ and $|\bar{\psi}_p^2\rangle$ is the same as before, because the evolution $S \to P$ is linear, *i.e.* projective.

Alternatively, it can be argued that the path $P \overset{(1)}{\rightarrow} S \overset{(2)}{\rightarrow} P$ is a *closed loop*, with a total phase shift which is the same interference angle. In fact both arguments are identical; in their second form it has been also put forward by Aharonov and Anandan.

In conclusion, there is no paradox in the use of states as rays and vector representatives, as long as everything is fibre preserving = projective invariant.

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